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DISCUSSION OF INVERSE FUNCTIONS.

BY COOPER D. SCHMITT, A. M., PROFESSOR OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENN.

1°. Notation. If $\sin x = a$, we know that x can be found, and hence in order to express this value of x explicitly we write it $x = \sin^{-1} a$. The first expression means, the sine of the arc, x, is a, and "sine" is the subject, that is, our attention is called to that as being equal to a. The second expression has the arc, x, for its subject, and the second member of the equation is algebraic short hand, telling what arc we mean.

We read it, "x equals the arc whose sine is a," or more briefly, "arc-sine a," *i. e.* x equals arc-sine a.

The expression is sometimes read "x equals the inverse sine of a," or "the anti-sine of a."

Similarly, $\tan 45^\circ = 1$, becomes $45^\circ = \frac{1}{4}\pi = \tan^{-1} 1$,

$\cos 60^\circ = \frac{1}{2}$, becomes $60^\circ = \frac{1}{3}\pi = \cos^{-1}\frac{1}{2}$.

Also, $y = \tan^{-1} x$ can be written $\tan y = x$, and so on. The convention is, that what is inverse on one side is written direct on the other; thus $\sec^{-1} a = x$ becomes $\sec x = a$.

2°. Since $\tan x = \tan x$ we can say $\tan^{-1}(\tan x) = x$ or $\tan(\tan^{-1}) x = x$. This becomes self-evident when expressed at length. Thus, the tangent of the arc whose tangent is x is x.

Similarly, $\cos^{-1}(\cos a) = a$, $\sin \sin^{-1} b = b$. The two symbols are not said to cancel each other but to annul or neutralize each other.

3°. There are always two angles less than 360° whose functions equal a certain quantity, but for simplicity, we will always mean the smaller of the two. Thus $\tan^{-1} 1 = 45^\circ$ or 225°, since the tangent of either of these angles is 1. But 45° is taken unless for some special reason the larger angle is desired.

4°. Any inverse function can be converted into all the other inverse functions.

Thus $\sin^{-1}\frac{3}{5}$ becomes $\tan^{-1}\frac{3}{4}$ by constructing a triangle with hypotenuse 5 and perpendicular 3. From this we see that $\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} = \cos^{-1}\frac{4}{5} = \cot^{-1}\frac{4}{3}$, etc., etc.

Similarly $\cos^{-1}\frac{2}{3} = \sin^{-1}\frac{\sqrt{5}}{3} = \tan^{-1}\frac{\sqrt{5}}{2}$, etc., etc.

Similarly $\tan^{-1}\frac{5}{12} = \cot^{-1}\frac{12}{5} = \sin^{-1}\frac{5}{13} = \cos^{-1}\frac{12}{13}$.

5°. Any formula of trigonometry expressed in direct notation can be converted into a corresponding formula in the inverse notation.

Thus to convert $\sin 2x = 2\sin x\cos x \ldots (1)$.

Let $\sin x = a$. Then $\cos x = \sqrt{1 - a^2}$ and $x = \sin^{-1} a$.

(1) can be written according to the convention agreed upon in 1°.

$2x = \sin^{-1}(2\sin x\cos x)$. Whence $2\sin^{-1} a = \sin^{-1}(2a\sqrt{1 - a^2})$.

Similarly $\cos(x + y) = \cos x\cos y - \sin x\sin y \ldots (2)$.

Let $\cos x = a$, then $x = \cos^{-1} a$. Let $\cos y = b$, then $y = \cos^{-1} b$.

And (2) can be written $x+y=\cos^{-1}(\cos x\cos y-\sin x\sin y)$.

Whence $\cos^{-1}a+\cos^{-1}b=\cos^{-1}(ab+\sqrt{1-a^2}\sqrt{1-b^2})\ldots.(3)$, which is a formula for adding two inverse cosines and getting the result as a single inverse cosine.

$$\text{Again } \tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y} \text{ or } x+y=\tan^{-1}\left(\frac{\tan x+\tan y}{1-\tan x\tan y}\right)\ldots(4).$$

Let $\tan x=a$, $\tan y=\tan b$, and (4) becomes

$$\tan^{-1}a+\tan^{-1}b=\tan^{-1}\frac{a+b}{1-ab}\ldots.(5).$$

$$\text{Similarly, } \tan^{-1}a-\tan^{-1}b=\tan^{-1}\frac{a-b}{1+ab}\ldots.(6).$$

6°. From 4° we see that any inverse function can be expressed as an inverse tangent, and by these last two formulae, (5) and (6), we can add or subtract any two inverse tangents ; hence these two formulae are all that are necessary for combining any number of inverse functions.

$$\text{Thus, } \tan^{-1}\tfrac{1}{7}+\tan^{-1}\tfrac{1}{13}=\tan^{-1}\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{91}}=\tan^{-1}\tfrac{2}{9}\,;$$

$$\tan^{-1}\tfrac{3}{4}+\tan^{-1}\tfrac{4}{3}=\tan^{-1}\frac{\frac{3}{4}+\frac{4}{3}}{1-1}=\tan^{-1}\infty=90^\circ=\tfrac{1}{2}\pi\,;$$

$$2\tan^{-1}\tfrac{3}{4}=\tan^{-1}\tfrac{3}{4}+\tan^{-1}\tfrac{3}{4}=\tan^{-1}\frac{\frac{3}{4}+\frac{3}{4}}{1-\frac{9}{16}}=\tan^{-1}\tfrac{24}{7}.$$

To add $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}$ we might use (3), but it is better to convert the $\cos^{-1}$ into $\tan^{-1}$ as in 4°.

$$\text{Thus } \cos^{-1}\tfrac{4}{5}+\cos^{-1}\tfrac{12}{13}=\tan^{-1}\tfrac{3}{4}+\tan^{-1}\tfrac{5}{12}=\tan^{-1}\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{5}{16}}=\tan^{-1}\tfrac{56}{33}.$$

7°. If fractional coefficients occur they must be gotten rid of.

Thus $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}-\frac{1}{2}\cos^{-1}\frac{3}{5}$ becomes

$$\tfrac{1}{2}(2\tan^{-1}\tfrac{1}{4}+2\tan^{-1}\tfrac{2}{9}-\cos^{-1}\tfrac{3}{5})$$

$$=\tfrac{1}{2}(\tan^{-1}\tfrac{1}{4}+\tan^{-1}\tfrac{1}{4}+\tan^{-1}\tfrac{2}{9}+\tan^{-1}\tfrac{2}{9}-\tan^{-1}\tfrac{4}{3})$$

$$=\tfrac{1}{2}\left(\tan^{-1}\frac{\frac{1}{4}+\frac{1}{4}}{1-\frac{1}{16}}+\tan^{-1}\frac{\frac{2}{9}+\frac{2}{9}}{1-\frac{4}{81}}-\tan^{-1}\tfrac{4}{3}\right)$$

$$=\tfrac{1}{2}(\tan^{-1}\tfrac{8}{15}+\tan^{-1}\tfrac{36}{77}-\tan^{-1}\tfrac{4}{3})$$

$$=\tfrac{1}{2}\left(\tan^{-1}\frac{\frac{8}{15}+\frac{36}{77}}{1-\frac{8}{15}\cdot\frac{36}{77}}-\tan^{-1}\tfrac{4}{3}\right)=\tfrac{1}{2}\left(\tan^{-1}\tfrac{1156}{867}-\tan^{-1}\tfrac{4}{3}\right)$$

$$=\tfrac{1}{2}\left(\tan^{-1}\tfrac{4}{3}-\tan^{-1}\tfrac{4}{3}\right)=0.$$

$4\tan^{-1}\frac{1}{5}$ can be written $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{5}$ and now the addition formulae can be used at once.

Similarly, for any integral coefficients.

Suppose we are to show that $\cos^{-1}\frac{63}{65}+2\tan^{-1}\frac{1}{5}=\sin^{-1}\frac{3}{5}$.

We proceed as follows: $\cos^{-1}\frac{63}{65}=\tan^{-1}\frac{16}{63}$, $\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$, and now we have to show that $\tan^{-1}\frac{16}{63}+2\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{3}{4}$, which is done at once as in previous example.

It is thus seen that the formulae $\tan^{-1}a\pm\tan^{-1}b=\tan^{-1}\dfrac{a\pm b}{1\mp ab}$ will apply to most if not all of the examples as given in our text-books and that the work is done entirely in the inverse notation.

CALCULUS.

84. Proposed by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.

Find the equation to the curve upon which a given ellipse must roll in order that one of its foci may describe a straight line.

No solution of this problem has yet been received.

It can be easily shown that $ds=(1+\frac{\rho}{\rho'})ds'$ where ρ and ρ' are the radii of curvature of the rolling and fixed curves, and ds and ds' arcs of the roulette (the curve generated by the focus of the ellipse) and the pedal curve, the origin of which is the generating point. Finding expressions for ds and ds' and substituting in the above equation and solving for ρ, we have an expression for the curvature of the fixed curve. By substituting for ρ, the general value for the radius of curvature, we derive a very complicated differential equation of the curve. I am not aware that the problem has ever been solved. If any of our contributors will send us a complete solution we shall be pleased to publish it. Ed. F.

85. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A line of double curvature, beginning at some point in the circumference of the base circle of a right cone, winds itself under the constant inclination β to the base circle around the curved surface of the cone. Find its length and that of its projection upon the base circle.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let ABC be the cone, CPG the line of curvature, σ its length, CE its projection on base, s the length of this projection, BO the z-axis. Also let $DO=CO=a$, $PF=r$, $\angle DPE=$semi-vertical angle of cone$=\gamma$, $\angle PCD=\beta$. Let P, C be